

# L<sup>p</sup>-SOLUTIONS OF A NONLINEAR THIRD ORDER DIFFERENTIAL EQUATION AND ASYMPTOTIC BEHAVIOR OF LINEAR FOURTH ORDER DIFFERENTIAL EQUATIONS

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**ABSTRACT.** In this paper we prove the well-posedness and we study the asymptotic behavior of nonoscillatory  $L^p$ -solutions for a third order nonlinear scalar differential equation. The equation consists of two parts: a linear third order with constant coefficients part and a nonlinear part represented by a polynomial of fourth order in three variables with variable coefficients. The results are obtained assuming three hypotheses: (i) the characteristic polynomial associated with the linear part has simple and real roots, (ii) the coefficients of the polynomial satisfy asymptotic integral smallness conditions, and (iii) the polynomial coefficients are in  $L^p([t_0, \infty[)$ . These results are applied to study a fourth order linear differential equation of Poincaré type and a fourth order linear differential equation with unbounded coefficients. Moreover, we give some examples where the classical theorems can not be applied.

## 1. INTRODUCTION

The study of asymptotic behavior for linear ordinary differential equations has been achieved their breakthroughs in the twentieth century, see [7] for a short historical review of the main landmarks, see also [4–6, 10, 12]. In [7] is revised the contributions of Poincaré [32], Perron [27], Levinson [25], Hartman-Wintner [21] and Harris and Lutz [18, 19]. Now, it is undisputed in the recent years the increasing interest in the development of the theory for the asymptotic analysis of high order linear differential equations, see for instance [7, 14–16, 30, 34, 35]. The main common motivation of these works come from two sources: the real world applications and the deepening of the theory. In particular, in the scalar method there is an interesting problem of the second kind, where the analysis of the asymptotic behavior of a linear equation is obtained via the analysis of a particular nonlinear equation. Thus, a natural question is to study the general form of that nonlinear equation.

In this paper we are interested in the study of  $L^p$ -solutions and the asymptotic behavior of the following third order nonlinear differential equation

$$z'''(t) + b_2 z''(t) + b_1 z'(t) + b_0 z(t) = \mathbb{P}(t, z(t), z'(t), z''(t)), \quad (1.1a)$$

where  $b_i$  are real constants and  $\mathbb{P} : \mathbb{R}^4 \rightarrow \mathbb{R}$  is a given function such that  $\mathbb{P}(x_0, x_1, x_2, x_3)$  is a polynomial of fourth degree in the three variables  $(x_1, x_2, x_3)$  and its coefficients depends on  $x_0$ , i.e.  $\mathbb{P}$  admits the representation

$$\mathbb{P}(x_0, \mathbf{x}) = \sum_{|\alpha|=0}^4 \Omega_{\alpha}(x_0) \mathbf{x}^{\alpha}, \quad \text{with } \mathbf{x} = (x_1, x_2, x_3), \quad \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \{0, 1, 2, 3, 4\}^3. \quad (1.1b)$$

Here, we have used the notation  $\{0, 1, 2, 3, 4\}^3$  for the cartesian product and  $|\alpha|$  and  $\mathbf{x}^{\alpha}$  for the standard multindex notation, i.e.  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$  and  $\mathbf{x}^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ .

The analysis of (1.1) is mainly motivated by the application of the scalar method to study the asymptotic behavior of nonoscillatory solutions for a fourth order linear differential equations of Poincaré type:

$$y^{(iv)} + [a_3 + r_3(t)]y''' + [a_2 + r_2(t)]y'' + [a_1 + r_1(t)]y' + [a_0 + r_0(t)]y = 0, \quad (1.2)$$

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where  $a_i$  are constants and  $r_i$  are real-valued functions. Indeed, we recall that the scalar method consists of three big steps: (i) a change of variable of the type  $z(t) = y'(t)[y(t)]^{-1} - \mu$  with  $\mu$  a characteristic root associated to (1.2) when  $r_0 = r_1 = r_2 = r_3 = 0$ , (ii) the analysis of existence, uniqueness and asymptotic behavior of the new nonlinear equation in terms of  $z$  which is given by

$$\begin{aligned} & z'''(t) + [4\mu + a_3]z''(t) + [6\mu^2 + 3a_3\mu + a_2]z'(t) + [4\mu^3 + 3\mu^2a_3 + 2\mu a_2 + a_1]z(t) \\ &= -\left\{ r_3(t)z''(t) + [3\mu r_3(t) + r_2(t)]z'(t) + [3\mu^2 r_3(t) + 2\mu r_2(t) + r_1(t)]z(t) \right. \\ &\quad + \mu^3 r_3(t) + \mu^2 r_2(t) + \mu r_1(t) + r_0(t) + 4z(t)z''(t) + [12\mu + 3a_3 + 3r_3(t)]z(t)z'(t) \\ &\quad + 6z(t)^2 z'(t) + 3[z'(t)]^2 + [6\mu^2 + 3\mu a_3 + a_2 + 3\mu r_3(t) + r_2(t)]z(t)^2 \\ &\quad \left. + [4\mu + r_3(t)]z(t)^3 + z(t)^4 \right\}, \end{aligned} \quad (1.3)$$

and (iii) the translation of the results from  $z$  to the original variable via the relation  $y(t) = \exp\left(\int_{t_0}^t (z(s) + \mu)ds\right)$ . Now, we note that the equation (1.3) is of the type (1.1).

Recently, in [7] the analysis of (1.3) in the context of  $C_0^2$ -solutions was developed by assuming three hypotheses. The first hypothesis is related to the constant coefficients  $a_i$  and set that the characteristic polynomial associated with the homogeneous equation for (1.3) (i.e. when  $r_0 = r_1 = r_2 = r_3 = 0$ ) has simple and real roots. The other two hypotheses are related to the behavior of the perturbation functions  $r_i$  and establish asymptotic integral smallness conditions of the perturbations. Now, under these general hypotheses and by application of a fixed point argument the authors prove that (1.3) has a unique solution in

$$C_0^2([t_0, \infty]) = \left\{ z \in C^2([t_0, \infty], \mathbb{R}) \quad : \quad z, z', z'' \rightarrow 0 \text{ when } t \rightarrow \infty \right\}, \quad t_0 \in \mathbb{R}$$

and also obtain the asymptotic behavior of the solutions for (1.3). Moreover, they establish the existence of a fundamental system of solutions and precise the formulas for the asymptotic behavior of the linear fourth order differential equation (1.2).

In this paper we prove the existence, uniqueness and asymptotic behavior for (1.1). Our results improve our previous results obtained in [7] for (1.3), since the equation (1.3) is a particular case of the equation (1.1). More precisely, we deduce the well posedness in  $C_0^2([t_0, \infty])$  by assuming that

- The roots  $\gamma_1, \gamma_2$  and  $\gamma_2$  of the corresponding characteristic polynomial associated to  $z''' + b_2 z'' + b_1 z' + b_0 z = 0$  (the homogeneous part of (1.1)) are real and simple.
- The coefficients  $\Omega_\alpha$  satisfy the following requirements

$$\begin{aligned} & \lim_{t \rightarrow \infty} \left| \int_{t_0}^{\infty} g(t, s) \Omega_\alpha(s) ds \right| + \left| \int_{t_0}^{\infty} \frac{\partial g}{\partial t}(t, s) \Omega_\alpha(s) ds \right| + \left| \int_{t_0}^{\infty} \frac{\partial^2 g}{\partial t^2}(t, s) \Omega_\alpha(s) ds \right| = 0, \\ & \lim_{t \rightarrow \infty} \int_{t_0}^{\infty} \left[ |g(t, s)| + \left| \frac{\partial g}{\partial t}(t, s) \right| + \left| \frac{\partial^2 g}{\partial t^2}(t, s) \right| \right] \left| \sum_{|\alpha|=1} \Omega_\alpha(s) \right| ds = 0, \\ & \sum_{k=1}^4 \int_{t_0}^{\infty} \left[ |g(t, s)| + \left| \frac{\partial g}{\partial t}(t, s) \right| + \left| \frac{\partial^2 g}{\partial t^2}(t, s) \right| \right] \left| \sum_{|\alpha|=k} \Omega_\alpha(s) \right| ds \quad \text{is bounded when } t \rightarrow \infty. \end{aligned}$$

Here  $g$  is a Green function.

Now, considering that

$$\sum_{|\alpha|=1}^4 |\Omega_\alpha(s)| \leq \rho \quad \text{for } \rho \in \left] 0, \frac{1}{\sigma \tilde{A}} \right[ ,$$

with  $\sigma$  and  $\hat{A}$  a given numbers in terms of  $\gamma_1, \gamma_2, \gamma_3$ , we get that

$$z(t), z'(t), z''(t) = \begin{cases} O\left(\int_t^\infty e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds\right), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{---}^3, \beta \in ]\gamma_1, 0[, \\ O\left(\int_{t_0}^\infty e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds\right), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+--}^3, \beta \in ]\gamma_2, 0[, \\ O\left(\int_{t_0}^\infty e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds\right), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{++-}^3, \beta \in ]\gamma_3, 0[, \\ O\left(\int_{t_0}^t e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds\right), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+++}^3, \beta \in ]0, \gamma_3[, \end{cases}$$

which represents the asymptotic behavior of the solutions for (1.1). Moreover, assuming that

- The coefficients  $\Omega_\alpha$  of  $\mathbb{P}$  are such that  $\Omega_\alpha \in L^p([t_0, \infty[)$  for  $|\alpha| = 0$  and for  $|\alpha| \geq 1$  the functions  $\Omega_\alpha$  are of the following type  $\Omega_\alpha(t) = \lambda_{\alpha,p} \Omega_{\alpha,p}(t) + \lambda_{\alpha,c}$ , where  $\lambda_{\alpha,p}$  and  $\lambda_{\alpha,c}$  are real constants and  $\Omega_{\alpha,p} \in L^p([t_0, \infty[)$ ,

we prove that the solution of (1.1) is a function belongs of the Sobolev space  $W^{2,p}([t_0, \infty[)$  and there exists  $m+1$  functions with  $m$  depending of  $p$  denoted by  $\Theta_1, \dots, \Theta_m$  and  $\Psi$ , such that  $\Theta_k \in W^{2,p/k}([t_0, \infty[)$ , for  $k = 1, \dots, m$ ,  $\Psi \in W^{2,1}([0, \infty[)$  and  $z^{(\ell)}(t) = \sum_{k=1}^m \Theta_k^{(\ell)}(t) + \Psi^{(\ell)}(t)$  for  $\ell = 0, 1, 2$ .

On the other side, in this paper we consider two applications of the results. First, assuming the hypotheses given in [7] for  $a_i$  and  $r_i$  and using the fact that perturbation functions  $r_i \in L^p([t_0, \infty[)$  we deduce the equation (1.3) has a solution belongs to the Sobolev space  $W^{2,p}([t_0, \infty[)$ . Moreover, we obtain a Levinson and Hartman-Wintner type results for the asymptotic behavior of the solutions for (1.2). The second application is the study of the asymptotic behavior of the solutions for the following fourth order differential equation

$$y^{(iv)}(t) - 2[q(t)]^{1/2} y'''(t) - q(t) y''(t) + 2[q(t)]^{3/2} y'(t) + r(t) y(t) = 0, \quad (1.4)$$

where  $q$  and  $r$  are real valued unbounded functions. In this case, the analysis is based in a change of variable, which transforms (1.4) in an equation of the type (1.2).

The paper is organized as follows. In section 2, we develop the analysis of existence, uniqueness and asymptotic behavior of (1.1) in  $C_0^2([t_0, \infty[)$ . Now, on section (3) we study the integrability of the solutions for equation (1.1) and we obtain a result for the asymptotic behavior of  $L^p$ -solutions. In section 4 we present the applications of the results to the analysis of (1.2) and (1.4). Finally, on section 5 we give some examples.

## 2. ANALYSIS OF EQUATION (1.1) IN $C_0^2([t_0, \infty[)$

In this section we present the results of well posedness and the asymptotic behavior of (1.1).

**2.1. Existence and uniqueness of (1.1).** Before to present the result of this subsection, we need to define some notations about Green functions. First, let us consider the equation associated to (1.1a) with  $\mathbb{P} = 0$ , i.e.

$$z''' + b_2 z'' + b_1 z' + b_0 z = 0, \quad (2.1)$$

and denote by  $\gamma_i$ ,  $i = 1, 2, 3$ , the roots of the characteristic polynomial for (2.1). Then, the Green function for (2.1) is defined by

$$g(t, s) = \frac{1}{(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)(\gamma_2 - \gamma_1)} \begin{cases} g_1(t, s), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{---}^3, \\ g_2(t, s), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+--}^3, \\ g_3(t, s), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{++-}^3, \\ g_4(t, s), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+++}^3, \end{cases} \quad (2.2)$$

where

$$g_1(t, s) = \begin{cases} 0, & t \geq s, \\ (\gamma_2 - \gamma_3)e^{-\gamma_1(t-s)} + (\gamma_3 - \gamma_1)e^{-\gamma_2(t-s)} + (\gamma_1 - \gamma_2)e^{-\gamma_3(t-s)}, & t \leq s, \end{cases} \quad (2.3)$$

$$g_2(t, s) = \begin{cases} (\gamma_2 - \gamma_3)e^{-\gamma_1(t-s)}, & t \geq s, \\ (\gamma_1 - \gamma_2)e^{-\gamma_3(t-s)} + (\gamma_3 - \gamma_1)e^{-\gamma_2(t-s)}, & t \leq s, \end{cases} \quad (2.4)$$

$$g_3(t, s) = \begin{cases} (\gamma_2 - \gamma_3)e^{-\gamma_1(t-s)} + (\gamma_3 - \gamma_1)e^{-\gamma_2(t-s)}, & t \geq s, \\ (\gamma_2 - \gamma_1)e^{-\gamma_3(t-s)}, & t \leq s, \end{cases} \quad (2.5)$$

$$g_4(t, s) = \begin{cases} (\gamma_2 - \gamma_3)e^{-\gamma_1(t-s)} + (\gamma_3 - \gamma_1)e^{-\gamma_2(t-s)} + (\gamma_1 - \gamma_2)e^{-\gamma_3(t-s)}, & t \geq s, \\ 0, & t \leq s. \end{cases} \quad (2.6)$$

Further details on Green functions may be consulted in [4]. Moreover, given  $g$  by (2.2), we define the functionals  $\mathcal{G}$  and  $\mathcal{L}$  as follows

$$\mathcal{G}(E)(t) = \left| \int_{t_0}^{\infty} g(t, s)E(s)ds \right| + \left| \int_{t_0}^{\infty} \frac{\partial g}{\partial t}(t, s)E(s)ds \right| + \left| \int_{t_0}^{\infty} \frac{\partial^2 g}{\partial t^2}(t, s)E(s)ds \right|, \quad (2.7)$$

$$\mathcal{L}(E)(t) = \int_{t_0}^{\infty} \left[ |g(t, s)| + \left| \frac{\partial g}{\partial t}(t, s) \right| + \left| \frac{\partial^2 g}{\partial t^2}(t, s) \right| \right] |E(s)|ds. \quad (2.8)$$

Note that the inequality  $0 \leq \mathcal{G}(E)(t) \leq \mathcal{L}(E)(t)$  holds for all  $t \geq t_0$ .

**Theorem 2.1.** *Let us introduce the notation  $C_0^2([t_0, \infty[)$  for the following space of functions*

$$C_0^2([t_0, \infty[) = \left\{ z \in C^2([t_0, \infty[, \mathbb{R}) : z, z', z'' \rightarrow 0 \text{ when } t \rightarrow \infty \right\}, \quad t_0 \in \mathbb{R},$$

*and consider the equation (1.1) where the constants  $b_i$  and the coefficients of  $\mathbb{P}$  satisfy the following two restrictions*

- (P<sub>1</sub>) *The roots  $\gamma_1, \gamma_2$  and  $\gamma_3$  of the corresponding characteristic polynomial associated to (2.1), the homogeneous part of (1.1a), are real and simple.*
- (P<sub>2</sub>) *The coefficients  $\Omega_\alpha$  of  $\mathbb{P}$  are such that*

$$\begin{aligned} \mathcal{G} \left( \sum_{|\alpha|=0} \Omega_\alpha \right) (t) &\rightarrow 0, \quad \mathcal{L} \left( \sum_{|\alpha|=1} |\Omega_\alpha| \right) (t) \rightarrow 0 \quad \text{and} \\ \sum_{k=1}^4 \mathcal{L} \left( \sum_{|\alpha|=k} |\Omega_\alpha| \right) (t) &\text{ bounded when } t \rightarrow \infty. \end{aligned}$$

*Here  $\mathcal{G}$  and  $\mathcal{L}$  are the operators defined on (2.7) and (2.8).*

*hold. Then, there is a unique  $z \in C_0^2([t_0, \infty[)$  solution of (1.1).*

*Proof.* By the method of variation of parameters, the hypothesis (P<sub>1</sub>), implies that the equation (1.1a) is equivalent to the following integral equation

$$z(t) = \int_{t_0}^{\infty} g(t, s) \mathbb{P}(s, z(s), z'(s), z''(s)) ds, \quad (2.9)$$

where  $g$  is the Green function defined on (2.2). Moreover, we recall that  $C_0^2([t_0, \infty[)$  is a Banach space with the norm  $\|z\|_0 = \sup_{t \geq t_0} [|z(t)| + |z'(t)| + |z''(t)|]$ . Now, we define the operator  $T$  from  $C_0^2([t_0, \infty[)$  to  $C_0^2([t_0, \infty[)$  as follows

$$Tz(t) = \int_{t_0}^{\infty} g(t, s) \mathbb{P}(s, z(s), z'(s), z''(s)) ds. \quad (2.10)$$

Then, we note that (2.9) can be rewritten as the operator equation

$$Tz = z \quad \text{over} \quad D_\eta := \left\{ z \in C_0^2([t_0, \infty[) : \|z\|_0 \leq \eta \right\}, \quad (2.11)$$

where  $\eta \in \mathbb{R}^+$  will be selected in order to apply the Banach fixed point theorem. Indeed, we have that

(a)  $T$  is well defined from  $C_0^2([t_0, \infty[)$  to  $C_0^2([t_0, \infty[)$ . Let us consider an arbitrary  $z \in C_0^2([t_0, \infty[)$ . We note that

$$\begin{aligned} T'z(t) &= \int_{t_0}^{\infty} \frac{\partial g}{\partial t}(t, s) \mathbb{P}(s, z(s), z'(s), z''(s)) ds, \\ T''z(t) &= \int_{t_0}^{\infty} \frac{\partial^2 g}{\partial t^2}(t, s) \mathbb{P}(s, z(s), z'(s), z''(s)) ds. \end{aligned}$$

Then, by the definition of  $g$ , we immediately deduce that  $Tz, T'z, T''z \in C^2([t_0, \infty[, \mathbb{R})$ . Furthermore, by (1.1b), we can deduce the following estimates

$$\begin{aligned} |z(t)| &\leq \left| \int_{t_0}^{\infty} g(t, s) \sum_{|\alpha|=0} \Omega_{\alpha}(s) ds \right| + \int_{t_0}^{\infty} |g(t, s)| \sum_{|\alpha|=1} |\Omega_{\alpha}(s)| |z(s)|^{\alpha_1} |z'(s)|^{\alpha_2} |z''(s)|^{\alpha_3} ds \\ &\quad + \int_{t_0}^{\infty} |g(t, s)| \sum_{|\alpha|=2}^4 |\Omega_{\alpha}(s)| |z(s)|^{\alpha_1} |z'(s)|^{\alpha_2} |z''(s)|^{\alpha_3} ds, \end{aligned} \quad (2.12)$$

$$\begin{aligned} |z'(t)| &\leq \left| \int_{t_0}^{\infty} \frac{\partial g}{\partial t}(t, s) \sum_{|\alpha|=0} \Omega_{\alpha}(s) ds \right| + \int_{t_0}^{\infty} \left| \frac{\partial g}{\partial t}(t, s) \right| \sum_{|\alpha|=1} |\Omega_{\alpha}(s)| |z(s)|^{\alpha_1} |z'(s)|^{\alpha_2} |z''(s)|^{\alpha_3} ds \\ &\quad + \int_{t_0}^{\infty} \left| \frac{\partial g}{\partial t}(t, s) \right| \sum_{|\alpha|=2}^4 |\Omega_{\alpha}(s)| |z(s)|^{\alpha_1} |z'(s)|^{\alpha_2} |z''(s)|^{\alpha_3} ds, \end{aligned} \quad (2.13)$$

$$\begin{aligned} |z''(t)| &\leq \left| \int_{t_0}^{\infty} \frac{\partial^2 g}{\partial t^2}(t, s) \sum_{|\alpha|=0} \Omega_{\alpha}(s) ds \right| + \int_{t_0}^{\infty} \left| \frac{\partial^2 g}{\partial t^2}(t, s) \right| \sum_{|\alpha|=1} |\Omega_{\alpha}(s)| |z(s)|^{\alpha_1} |z'(s)|^{\alpha_2} |z''(s)|^{\alpha_3} ds \\ &\quad + \int_{t_0}^{\infty} \left| \frac{\partial^2 g}{\partial t^2}(t, s) \right| \sum_{|\alpha|=2}^4 |\Omega_{\alpha}(s)| |z(s)|^{\alpha_1} |z'(s)|^{\alpha_2} |z''(s)|^{\alpha_3} ds. \end{aligned} \quad (2.14)$$

Now, by application of the hypothesis  $(P_2)$ , we have that the right hand sides of (2.12)-(2.14) tend to 0 when  $t \rightarrow \infty$ . Then,  $Tz, T'z, T''z \rightarrow 0$  when  $t \rightarrow \infty$  or equivalently  $Tz \in C_0^2$  for all  $z \in C_0^2$ .

(b) For all  $\eta \in ]0, 1[$ , the set  $D_{\eta}$  is invariant under  $T$ . Let us consider  $z \in D_{\eta}$ . From (2.12)-(2.14), we get the following estimate

$$\|Tz\|_0 \leq \mathcal{G} \left( \sum_{|\alpha|=0} \Omega_{\alpha} \right) (t) + \sum_{k=1}^4 \|z\|_0^k \mathcal{L} \left( \sum_{|\alpha|=k} |\Omega_{\alpha}| \right) (t). \quad (2.15)$$

Now, by  $(P_2)$  we deduce that the first term on the right hand side of (2.15) converges to 0 when  $t \rightarrow \infty$ . Similarly, by application of  $(P_2)$ , we can prove that the inequality

$$\begin{aligned} &\sum_{k=1}^4 \|z\|_0^k \mathcal{L} \left( \sum_{|\alpha|=k} |\Omega_{\alpha}| \right) (t) \\ &\leq \eta^2 \left\{ \mathcal{L} \left( \sum_{|\alpha|=2} |\Omega_{\alpha}| \right) (t) + \eta \mathcal{L} \left( \sum_{|\alpha|=3} |\Omega_{\alpha}| \right) (t) + \eta^2 \mathcal{L} \left( \sum_{|\alpha|=4} |\Omega_{\alpha}| \right) (t) \right\} \\ &\leq \eta \end{aligned}$$

holds when  $t \rightarrow \infty$  in a right neighborhood of  $\eta = 0$ . Hence, by (2.15) and  $(P_2)$ , we prove that  $Tz \in D_{\eta}$  for all  $z \in D_{\eta}$ .

(c)  $T$  is a contraction for  $\eta \in ]0, 1[$ . Let  $z_1, z_2 \in D_\eta$ , by (1.1b) and algebraic rearrangements, we follow that

$$\begin{aligned} \|Tz_1 - Tz_2\|_0 &\leq \sum_{k=1}^4 \|z_1 - z_2\|_0^k \mathcal{L} \left( \sum_{|\alpha|=k} |\Omega_\alpha| \right) (t) \\ &\leq \|z_1 - z_2\|_0 \max \{1, \eta, \eta^2, \eta^3\} \sum_{k=1}^4 \mathcal{L} \left( \sum_{|\alpha|=k} |\Omega_\alpha| \right) (t). \end{aligned}$$

Then, by application of  $(P_2)$ , we deduce that  $T$  is a contraction, since, for an arbitrary  $\eta \in ]0, 1[$ , we have that  $\max \{1, \eta, \eta^2, \eta^3\} = \eta < 1$ .

Hence, from (a)-(c) and application of Banach fixed point theorem, we deduce that there is a unique  $z \in D_\eta \subset C_0^2([t_0, \infty])$  solution of (2.11).  $\square$

**2.2. Asymptotic behavior of the solution for (1.1).** Before to present the result of this subsection, we deduce a useful bound of the Green function (2.2). To fix ideas, we consider  $g_1$  and we have that

$$\begin{aligned} |g_1(t, s)| + \left| \frac{\partial g_1}{\partial t}(t, s) \right| + \left| \frac{\partial^2 g_1}{\partial t^2}(t, s) \right| &\leq (|\gamma_3 - \gamma_2| + |\gamma_1 - \gamma_3| + |\gamma_2 - \gamma_1|) e^{-\max \{\gamma_1, \gamma_2, \gamma_3\}(t-s)}, \\ &+ (|\gamma_3 - \gamma_2||\gamma_1| + |\gamma_1 - \gamma_3||\gamma_2| + |\gamma_2 - \gamma_1||\gamma_3|) e^{-\max \{\gamma_1, \gamma_2, \gamma_3\}(t-s)}, \\ &+ (|\gamma_3 - \gamma_2||\gamma_1|^2 + |\gamma_1 - \gamma_3||\gamma_2|^2 + |\gamma_2 - \gamma_1||\gamma_3|^2) e^{-\max \{\gamma_1, \gamma_2, \gamma_3\}(t-s)} \\ &\leq A e^{-\gamma_1(t-s)}, \end{aligned} \quad (2.16)$$

where

$$A = |\gamma_3 - \gamma_2|(1 + |\gamma_1| + |\gamma_1|^2) + |\gamma_3 - \gamma_1|(1 + |\gamma_2| + |\gamma_2|^2) + |\gamma_2 - \gamma_1|(1 + |\gamma_3| + |\gamma_3|^2). \quad (2.17)$$

Analogously, we can prove the bounds for  $i = 2, 3$ , and in general we obtain that

$$|g_i(t, s)| + \left| \frac{\partial g_i}{\partial t}(t, s) \right| + \left| \frac{\partial^2 g_i}{\partial t^2}(t, s) \right| \leq A e^{-\gamma_i(t-s)}, \quad i = 1, 2, 3. \quad (2.18)$$

**Theorem 2.2.** Consider that the hypotheses of Theorem 2.1 are satisfied and assume that  $\gamma_1 > \gamma_2 > \gamma_3$ . Moreover consider the positive number  $\sigma$  depending of  $\gamma_1, \gamma_2, \gamma_3$  and a given number  $\beta$  defined as follows

$$\sigma = \begin{cases} \frac{1}{-\gamma_1 + \beta}, & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{---}^3 \quad \text{and} \quad \beta \in ]\gamma_1, 0[, \\ \frac{1}{-(-\gamma_1 + \beta)} + \frac{1}{-\gamma_2 + \beta}, & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+--}^3 \quad \text{and} \quad \beta \in ]\gamma_2, 0[, \\ \frac{1}{-(-\gamma_2 + \beta)} + \frac{1}{-\gamma_3 + \beta}, & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{++-}^3 \quad \text{and} \quad \beta \in ]\gamma_3, 0[, \\ \frac{1}{-(-\gamma_3 + \beta)}, & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+++}^3 \quad \text{and} \quad \beta \in ]0, \gamma_3[. \end{cases} \quad (2.19)$$

If the coefficients of  $\mathbb{P}$  satisfies the following estimate

$$\sum_{|\alpha|=1}^4 |\Omega_\alpha(s)| \leq \rho \quad \text{for } \rho \in \left] 0, \frac{1}{\sigma \hat{A}} \right[ \quad \text{with} \quad \hat{A} = \frac{A}{|(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)|}, \quad (2.20)$$

where  $A$  defined on (2.17), then the solution of (1.1) has the following asymptotic behavior

$$z(t), z'(t), z''(t) = \begin{cases} O\left(\int_t^\infty e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds\right), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{---}^3, \beta \in ]\gamma_1, 0[, \\ O\left(\int_{t_0}^\infty e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds\right), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+--}^3, \beta \in ]\gamma_2, 0[, \\ O\left(\int_{t_0}^\infty e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds\right), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{++-}^3, \beta \in ]\gamma_3, 0[, \\ O\left(\int_{t_0}^t e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds\right), & (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+++}^3, \beta \in ]0, \gamma_3[. \end{cases} \quad (2.21)$$

*Proof.* We prove the formula (2.21) by analyzing an iterative sequence and using the properties of the operator  $T$  defined in (2.10).

*Proof of (2.21) for  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{---}^3$ .* Let us denote by  $T$  the operator defined in (2.10). Now, on  $D_\eta$  with  $\eta \in ]0, 1[$ , we define the sequence  $\omega_{n+1} = T\omega_n$  with  $\omega_0 = 0$ , we have that  $\omega_n \rightarrow z$  when  $n \rightarrow \infty$ . This fact is a consequence of the contraction property of  $T$ .

We note that the Green function  $g$  defined on (2.2) is given in terms of  $g_1$ , since  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{---}^3$ . Then, we have that the operator  $T$  can be rewritten equivalently as follows

$$Tz(t) = \frac{1}{(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)} \int_t^\infty g_1(t, s) \mathbb{P}\left(s, z(s), z'(s), z''(s)\right) ds, \quad \text{for } t \geq t_0, \quad (2.22)$$

since  $g_1(t, s) = 0$  for  $s \in [t_0, t]$ . Thus, the proof of (2.21) is reduced to prove that

$$\exists \Phi_n \in \mathbb{R}_+ : |\omega_n(t)| + |\omega'_n(t)| + |\omega''_n(t)| \leq \Phi_n \int_t^\infty e^{-\beta(t-\tau)} \left| \sum_{|\alpha|=0} \Omega_\alpha(\tau) \right| d\tau, \quad \forall t \geq t_0, \quad (2.23)$$

$$\exists \Phi \in \mathbb{R}_+ : \Phi_n \rightarrow \Phi, \text{ when } n \rightarrow \infty. \quad (2.24)$$

Hence, to complete the proof of (2.21) with  $i = 1$ , we proceed to prove (2.23) by mathematical induction on  $n$  and deduce that (2.24) is a consequence of the construction of the sequence  $\{\Phi_n\}$ .

We now prove (2.23). Note that for  $n = 1$  the estimate (2.23) is satisfied with  $\Phi_1 = \hat{A}$ . Indeed, it can be proved immediately by the definition of the operator  $T$  given on (2.22), the property  $\mathbb{P}(s, 0, 0, 0) = 0$ , the estimate (2.16) and the hypothesis that  $\beta \in [\gamma_1, 0[$ , since

$$\begin{aligned} |\omega_1(t)| + |\omega'_1(t)| + |\omega''_1(t)| &= |T\omega_0(t)| + |T'\omega_0(t)| + |T''\omega_0(t)| \\ &= \frac{1}{|(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)|} \\ &\quad \times \int_t^\infty \left( |g_1(t, s)| + \left| \frac{\partial g_1}{\partial t}(t, s) \right| + \left| \frac{\partial^2 g_1}{\partial t^2}(t, s) \right| \right) \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds \\ &\leq \frac{A}{|(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)|} \int_t^\infty e^{-\gamma_1(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds \\ &\leq \hat{A} \int_t^\infty e^{-\beta(t-\tau)} \left| \sum_{|\alpha|=0} \Omega_\alpha(\tau) \right| d\tau. \end{aligned}$$

Now, assuming that (2.23) is valid for  $n = k$ , we prove that (2.23) is also valid for  $n = k + 1$ . However, before to prove the estimate (2.23) for  $n = k + 1$ , we note that by (1.1b) and the fact that  $\max\{1, \eta, \eta^2, \eta^3\} = 1$ , we deduce the following estimate

$$\left| \mathbb{P}\left(s, \omega_k(s), \omega'_k(s), \omega''_k(s)\right) \right| \leq \sum_{|\alpha|=0}^4 \left| \Omega_\alpha(s) \right| \left| \omega_k(s) \right|^{\alpha_1} \left| \omega'_k(s) \right|^{\alpha_2} \left| \omega''_k(s) \right|^{\alpha_3}$$

$$\begin{aligned}
&\leq \sum_{|\alpha|=0} \left| \Omega_{\alpha}(s) \right| + \left( |\omega_k(s)| + |\omega'_k(s)| + |\omega''_k(s)| \right) \left( \sum_{|\alpha|=1}^4 \left| \Omega_{\alpha}(s) \right| \eta^{|\alpha|-1} \right) \\
&\leq \sum_{|\alpha|=0} \left| \Omega_{\alpha}(s) \right| + \left( |\omega_k(s)| + |\omega'_k(s)| + |\omega''_k(s)| \right) \sum_{|\alpha|=1}^4 \left| \Omega_{\alpha}(s) \right|. \tag{2.25}
\end{aligned}$$

Using (2.22), the inductive hypothesis, the inequality (2.16) and the estimate (2.25) we have that

$$\begin{aligned}
&|\omega_{k+1}(t)| + |\omega'_{k+1}(t)| + |\omega''_{k+1}(t)| \\
&= |T\omega_k(t)| + |T'\omega_k(t)| + |T''\omega_k(t)| \\
&= \frac{1}{|(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)|} \left\{ \left| \int_t^\infty g_1(t, s) \mathbb{P}(s, \omega_k(s), \omega'_k(s), \omega''_k(s)) ds \right| \right. \\
&\quad + \left| \int_t^\infty \frac{\partial g_1}{\partial t}(t, s) \mathbb{P}(s, \omega_k(s), \omega'_k(s), \omega''_k(s)) ds \right| \\
&\quad \left. + \left| \int_t^\infty \frac{\partial^2 g_1}{\partial t^2}(t, s) \mathbb{P}(s, \omega_k(s), \omega'_k(s), \omega''_k(s)) ds \right| \right\} \\
&= \frac{1}{|(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)|} \\
&\quad \times \int_t^\infty \left( |g_1(t, s)| + \left| \frac{\partial g_1}{\partial t}(t, s) \right| + \left| \frac{\partial^2 g_1}{\partial t^2}(t, s) \right| \right) \left| \mathbb{P}(s, \omega_k(s), \omega'_k(s), \omega''_k(s)) \right| ds \\
&\leq \hat{A} \int_t^\infty e^{-\gamma_1(t-s)} \left\{ \sum_{|\alpha|=0} \left| \Omega_{\alpha}(s) \right| + \left( |\omega_k(s)| + |\omega'_k(s)| + |\omega''_k(s)| \right) \sum_{|\alpha|=1}^4 \left| \Omega_{\alpha}(s) \right| \right\} ds \\
&\leq \hat{A} \int_t^\infty e^{-\gamma_1(t-s)} \left\{ \sum_{|\alpha|=0} \left| \Omega_{\alpha}(s) \right| + \sum_{|\alpha|=1}^4 \left| \Omega_{\alpha}(s) \right| \Phi_k \int_s^\infty e^{-\gamma_1(s-\tau)} \sum_{|\alpha|=0} \left| \Omega_{\alpha}(\tau) \right| d\tau \right\} ds \\
&\leq \hat{A} \left\{ 1 + \int_t^\infty e^{-\gamma_1(t-s)} \sum_{|\alpha|=1}^4 \left| \Omega_{\alpha}(s) \right| \Phi_k ds \right\} \int_t^\infty e^{-\gamma_1(t-\tau)} \sum_{|\alpha|=0} \left| \Omega_{\alpha}(\tau) \right| d\tau \\
&= \hat{A} \left( 1 + \frac{\Phi_k \rho}{-\gamma_1 + \beta} \right) \int_t^\infty e^{-\beta(t-\tau)} \sum_{|\alpha|=0} \left| \Omega_{\alpha}(\tau) \right| d\tau.
\end{aligned}$$

Then, by the induction process, (2.23) is satisfied with  $\Phi_n = \hat{A}(1 + \Phi_{n-1} \rho (-\gamma_1 + \beta)^{-1})$ . By (2.19) we can rewrite  $\Phi_n$  as follows  $\Phi_n = \hat{A}(1 + \Phi_{n-1} \rho \sigma)$

The proof of (2.24) is given as follows. Using recursively the definition of  $\Phi_{n-2}, \dots, \Phi_2$ , we can rewrite  $\Phi_n$  as the sum of the terms of a geometric progression where the common ratio is given by  $\rho \hat{A} \sigma$ . Then, the hypothesis (2.20) implies the existence of  $\Phi$  satisfying (2.24), since by the construction of  $\rho$  we have that  $\rho \hat{A} \sigma \in ]0, 1[$ . More precisely, we deduce that

$$\lim_{n \rightarrow \infty} \Phi_n = \hat{A} \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (\rho \hat{A} \sigma)^i = \hat{A} \lim_{n \rightarrow \infty} \frac{[(\rho \hat{A} \sigma)^n - 1]}{\rho \hat{A} |\gamma_1|^{-1} - 1} = \frac{\hat{A}}{1 - \rho \hat{A} \sigma} = \Phi > 0.$$

Hence, (2.23)-(2.24) are valid and the proof of (2.21) for  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{---}^3$  is concluded by passing to the limit the sequence  $\{\Phi_n\}$  when  $n \rightarrow \infty$  in the topology of  $C_0^2([t_0, \infty])$ .

*Proof of (2.21) for  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+--}^3$ .* Similarly to the case  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{---}^3$  we define the sequence  $\omega_{n+1} = T\omega_n$  with  $\omega_0 = 0$  and, by the contraction property of  $T$ , we can deduce that  $\omega_n \rightarrow z$  when  $n \rightarrow \infty$ . Then, the Green function  $g$  defined on (2.2) is given in terms of  $g_2$ .



Thereby, the operator  $T$  can be rewritten equivalently as follows

$$\begin{aligned}
Tz(t) &= \frac{1}{(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)} \int_{t_0}^{\infty} g_2(t, s) \mathbb{P}(s, \omega_k(s), \omega'_k(s), \omega''_k(s)) ds \\
&= \frac{1}{(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)} \left\{ \int_{t_0}^t (\gamma_2 - \gamma_3) e^{-\gamma_1(t-s)} \mathbb{P}(s, \omega_k(s), \omega'_k(s), \omega''_k(s)) ds \right. \\
&\quad \left. + \int_t^{\infty} [(\gamma_1 - \gamma_2) e^{-\gamma_3(t-s)} + (\gamma_3 - \gamma_1) e^{-\gamma_2(t-s)}] \mathbb{P}(s, \omega_k(s), \omega'_k(s), \omega''_k(s)) ds \right\}.
\end{aligned} \tag{2.26}$$

Then, the proof of (2.21) for  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+--}^3$  is reduced to prove

$$\exists \Phi_n \in \mathbb{R}_+ : |\omega_n(t)| + |\omega'_n(t)| + |\omega''_n(t)| \leq \Phi_n \int_{t_0}^{\infty} e^{-\beta(t-\tau)} \left| \sum_{|\alpha|=0} \Omega_{\alpha}(\tau) \right| d\tau, \quad \forall t \geq t_0, \tag{2.27}$$

$$\exists \Phi \in \mathbb{R}_+ : \Phi_n \rightarrow \Phi \text{ when } n \rightarrow \infty. \tag{2.28}$$

In the induction step for  $n = 1$  the estimate (2.27) is satisfied with  $\Phi_1 = \hat{A}$ , since by the definition of the operator  $T$  given on (2.26), the property  $\mathbb{P}(s, 0, 0, 0) = 0$ , the estimate (2.18) and the fact that  $\beta \in [\gamma_2, 0] \subset [\gamma_3, \gamma_1]$ , we deduce the following bound

$$\begin{aligned}
|\omega_1(t)| + |\omega'_1(t)| + |\omega''_1(t)| &= |T\omega_0(t)| + |T'\omega_0(t)| + |T''\omega_0(t)| \\
&\leq \frac{1}{|(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1)|} \\
&\quad \times \left\{ |\gamma_2 - \gamma_3| \left( 1 + |\gamma_1| + |\gamma_1|^2 \right) \int_{t_0}^t e^{-\gamma_1(t-s)} \left| \sum_{|\alpha|=0} \Omega_{\alpha}(\tau) \right| ds \right. \\
&\quad \left[ |\gamma_2 - \gamma_1| \left( 1 + |\gamma_3| + |\gamma_3|^2 \right) + |\gamma_3 - \gamma_1| \left( 1 + |\gamma_2| + |\gamma_2|^2 \right) \right] \\
&\quad \left. \times \int_t^{\infty} e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_{\alpha}(\tau) \right| ds \right\} \\
&\leq \hat{A} \left\{ \int_{t_0}^t e^{-\max\{\gamma_2, \gamma_3\}(t-s)} \left| \sum_{|\alpha|=0} \Omega_{\alpha}(s) \right| ds + \int_t^{\infty} e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_{\alpha}(s) \right| ds \right\} \\
&= \hat{A} \int_{t_0}^{\infty} e^{-\beta(t-\tau)} \left| \sum_{|\alpha|=0} \Omega_{\alpha}(\tau) \right| d\tau.
\end{aligned}$$

Then, the general induction step can be proved as follows

$$\begin{aligned}
&|\omega_{k+1}(t)| + |\omega'_{k+1}(t)| + |\omega''_{k+1}(t)| \\
&= |T\omega_k(t)| + |T'\omega_k(t)| + |T''\omega_k(t)| \\
&\leq \hat{A} \int_{t_0}^t e^{-\gamma_1(t-s)} \left\{ \left| \sum_{|\alpha|=0} \Omega_{\alpha}(s) \right| + (|\omega_k(s)| + |\omega'_k(s)| + |\omega''_k(s)|) \sum_{|\alpha|=1}^4 \left| \Omega_{\alpha}(s) \right| \right\} ds \\
&+ \hat{A} \int_t^{\infty} e^{-\gamma_2(t-s)} \left\{ \left| \sum_{|\alpha|=0} \Omega_{\alpha}(s) \right| + (|\omega_k(s)| + |\omega'_k(s)| + |\omega''_k(s)|) \sum_{|\alpha|=1}^4 \left| \Omega_{\alpha}(s) \right| \right\} ds \\
&= \hat{A} [J_1(t) + J_2(t)] \\
&\leq \hat{A} (1 + \Phi_k \rho \sigma) \int_{t_0}^{\infty} e^{-\beta(t-\tau)} \left| \sum_{|\alpha|=0} \Omega_{\alpha}(\tau) \right| d\tau,
\end{aligned}$$

where  $\sigma$  is the number defined on (2.19), since for  $\beta \in ]\gamma_2, 0[ \subset ]\gamma_3, \gamma_1[$ , we can deduce that

$$\begin{aligned}
J_1(t) &:= \int_{t_0}^t e^{-\gamma_1(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds + \int_t^\infty e^{-\gamma_2(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds \\
&\leq \int_{t_0}^t e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds + \int_t^\infty e^{-\beta(t-s)} \left| \sum_{|\alpha|=0} \Omega_\alpha(s) \right| ds \\
&= \int_{t_0}^\infty e^{-\beta(t-\tau)} \left| \sum_{|\alpha|=0} \Omega_\alpha(\tau) \right| d\tau, \\
J_2(t) &:= \int_{t_0}^t e^{-\gamma_1(t-s)} \left( |\omega_k(s)| + |\omega'_k(s)| + |\omega''_k(s)| \right) \sum_{|\alpha|=1}^4 \left| \Omega_\alpha(s) \right| ds \\
&\quad + \int_t^\infty e^{-\gamma_2(t-s)} \left( |\omega_k(s)| + |\omega'_k(s)| + |\omega''_k(s)| \right) \sum_{|\alpha|=1}^4 \left| \Omega_\alpha(s) \right| ds \\
&\leq \rho \int_{t_0}^t e^{-\gamma_1(t-s)} \left( |\omega_k(s)| + |\omega'_k(s)| + |\omega''_k(s)| \right) ds \\
&\quad + \rho \int_t^\infty e^{-\gamma_2(t-s)} \left( |\omega_k(s)| + |\omega'_k(s)| + |\omega''_k(s)| \right) ds \\
&\leq \Phi_k \rho \left( \frac{1 - \exp(-\sigma_1(t_0 - t))}{-(-\sigma_1 + \beta)} + \frac{1}{-\sigma_2 + \beta} \right) \int_{t_0}^\infty e^{-\beta(t-\tau)} \left| \sum_{|\alpha|=0} \Omega_\alpha(\tau) \right| d\tau \\
&\leq \Phi_k \rho \sigma \int_{t_0}^\infty e^{-\beta(t-\tau)} \left| \sum_{|\alpha|=0} \Omega_\alpha(\tau) \right| d\tau.
\end{aligned}$$

Hence the thesis of the inductive steps holds with  $\Phi_n = \hat{A}(1 + \Phi_{n-1}\rho\sigma)$ .

We proceed in an analogous way to the case  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{---}^3$  and deduce that (2.28) is satisfied with  $\Phi = \hat{A}/(1 - \rho\sigma\hat{A}) > 0$ .

Therefore, the sequence  $\{\Phi_n\}$  is convergent and  $z_2$  (the limit of  $\omega_n$  in the topology of  $C_0^2([t_0, \infty])$ ) satisfies (2.21).

*Proof of (2.21) for  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{++-}^3$  and for  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+++}^3$ .* The proof of (2.21) for  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{++-}^3$  and  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+++}^3$  are completely analogous to the proofs of (2.21) for  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{---}^3$  and  $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}_{+-+}^3$ , respectively.  $\square$

### 3. $L^p$ -SOLUTIONS FOR (1.1)

**Theorem 3.1.** *Let us consider that the hypotheses of Theorem 2.2 are satisfied and denote by  $\lceil \cdot \rceil$  the ceiling function and by  $W^{2,p}([t_0, \infty[)$  the Sobolev space defined by*

$$\lceil x \rceil = n + 1, \quad x \in ]n, n + 1], \quad n \in \mathbb{Z}, \quad \text{and} \quad (3.1)$$

$$W^{2,p}([t_0, \infty[) = \left\{ u \in L^p([t_0, \infty[) : u', u'' \in L^p([t_0, \infty[) \right\}, \quad (3.2)$$

respectively. Moreover assume that the condition

(P<sub>3</sub>) *The coefficients  $\Omega_\alpha$  of  $\mathbb{P}$  are such that  $\Omega_\alpha \in L^p([t_0, \infty[)$  for  $|\alpha| = 0$  and for  $|\alpha| \geq 1$  the functions  $\Omega_\alpha$  are of the following type  $\Omega_\alpha(t) = \lambda_{\alpha,p} \Omega_{\alpha,p}(t) + \lambda_{\alpha,c}$ , where  $\lambda_{\alpha,p}$  and  $\lambda_{\alpha,c}$  are real constants and  $\Omega_{\alpha,p} \in L^p([t_0, \infty[)$ .*

is satisfied. Then, the following assertions are valid for  $z$ , the solution of (1.1),

- (i)  $z$  is a function belongs to the Sobolev space  $W^{2,p}([t_0, \infty[)$ .
- (ii) Let  $m$  the number defined by  $m = \lceil p \rceil - 1$  for  $p \in ]1, 4]$  and by  $m = 4$  for  $p \in ]4, \infty]$ . There exists  $m + 1$  functions, denoted by  $\Theta_1, \dots, \Theta_m$  and  $\Psi$ , such that  $\Theta_k \in W^{2,p/k}([t_0, \infty[)$ , for  $k = 1, \dots, m$ ,  $\Psi \in W^{2,1}([0, \infty[)$  and  $z^{(\ell)}(t) = \sum_{k=1}^m \Theta_k^{(\ell)}(t) + \Psi^{(\ell)}(t)$  for  $\ell = 0, 1, 2$ .

*Proof.* (i). Let us denote by  $\gamma_\beta$  and  $\mathbb{Y}$  the functions defined as follows

$$\begin{aligned}\gamma_\beta(t) &= \begin{cases} \exp(-\beta t) & t \geq t_0, \\ 0 & \text{elsewhere,} \end{cases} \quad \text{for some } t_0 \geq 0, \\ \mathbb{Y}(t) &= \int_{t_0}^{\infty} \exp(-\beta(t-s)) \sum_{|\alpha|=0} \Omega_\alpha(s) ds.\end{aligned}$$

We note that  $\mathbb{Y} = \gamma_\beta * \sum_{|\alpha|=0} \Omega_\alpha$ , where  $*$  denotes the convolution. Then, by the convolution properties and the hypothesis that the coefficients  $\Omega_\alpha$  of  $\mathbb{P}$  with  $|\alpha| = 0$  are belong of  $L^p([t_0, \infty[)$ , we follow that  $\mathbb{Y}$  is belongs to  $L^p([t_0, \infty[)$ . Thus, by (2.21), we deduce that  $z, z', z'' \in L^p([t_0, \infty[)$  or equivalently by (3.2)  $z$  is belongs  $W^{2,p}([0, \infty[)$ .

(ii). By (2.10)-(2.11) and (1.1b), we have that  $z_i$  can be rewritten as follows

$$\begin{aligned}z(s) &= \int_{t_0}^{\infty} g(t, s) \mathbb{P}(s, z(s), z'(s), z''(s)) ds \\ &= \sum_{k=0}^4 \int_{t_0}^{\infty} g(t, s) \sum_{|\alpha|=k} \Omega_\alpha(s) [z(s)]^{\alpha_1} [z'(s)]^{\alpha_2} [z''(s)]^{\alpha_3} ds := \sum_{k=0}^4 \mathcal{J}_k(s).\end{aligned}\quad (3.3)$$

Now, we apply the hypothesis  $(P_3)$  to construct the functions  $\Theta^i$  and  $\Psi$ .

If  $|\alpha| = 0$ , by  $(P_3)$  and similar arguments to those used in the proof of item (i) we have that  $\mathcal{J}_0 \in W^{2,p}([t_0, \infty[)$ . Moreover, if  $|\alpha| = k \in \{1, 2, 3, 4\}$ , by application of  $(P_3)$  and item (i) we deduce that  $z, z', z'' \in W^{2,p}([t_0, \infty[)$ , which implies that  $[z]^{\alpha_1} [z']^{\alpha_2} [z'']^{\alpha_3} \in L^{p/k}([t_0, \infty[)$ . Now, by  $(P_3)$  for  $|\alpha| > 1$  we have that

$$\begin{aligned}\mathcal{J}_k(s) &= \int_{t_0}^{\infty} g(t, s) \sum_{|\alpha|=k} \left( \lambda_{\alpha,p} \Omega_{\alpha,p}(s) + \lambda_{\alpha,c} \right) [z(s)]^{\alpha_1} [z'(s)]^{\alpha_2} [z''(s)]^{\alpha_3} ds \\ &= \sum_{|\alpha|=k} \lambda_{\alpha,p} \int_{t_0}^{\infty} g(t, s) \Omega_{\alpha,p}(s) [z(s)]^{\alpha_1} [z'(s)]^{\alpha_2} [z''(s)]^{\alpha_3} ds \\ &\quad + \sum_{|\alpha|=k} \lambda_{\alpha,c} \int_{t_0}^{\infty} g(t, s) [z(s)]^{\alpha_1} [z'(s)]^{\alpha_2} [z''(s)]^{\alpha_3} ds \\ &:= \mathcal{J}_{k,p}(s) + \mathcal{J}_{k,c}(s),\end{aligned}\quad (3.4)$$

i.e.  $\mathcal{J}_k$  can be rewritten as the linear combination of the functions  $\mathcal{J}_{k,p} \in W^{2,p/(1+k)}([t_0, \infty[)$  and  $\mathcal{J}_{k,c} \in W^{2,p/k}([t_0, \infty[)$ .

By (3.3) and (3.4) we have that  $z$  can be discomposed

$$\begin{aligned}z(s) &= \mathcal{J}_0(s) + \sum_{k=1}^4 \mathcal{J}_{k,p}(s) + \sum_{k=1}^4 \mathcal{J}_{k,c}(s) \\ &= \left( [\mathcal{J}_0 + \mathcal{J}_{1,c}] + [\mathcal{J}_{1,p} + \mathcal{J}_{2,c}] + [\mathcal{J}_{2,p} + \mathcal{J}_{3,c}] + [\mathcal{J}_{3,p} + \mathcal{J}_{4,c}] + \mathcal{J}_{4,p} \right)(s), \\ &= \left( \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{J}_{4,p} \right)(s),\end{aligned}$$

with

$$\begin{aligned}\mathcal{H}_1 &:= \mathcal{J}_0 + \mathcal{J}_{1,c} \in W^{2,p}([t_0, \infty[), \quad \mathcal{H}_2 := \mathcal{J}_{1,p} + \mathcal{J}_{2,c} \in W^{2,p/2}([t_0, \infty[), \\ \mathcal{H}_3 &:= \mathcal{J}_{2,p} + \mathcal{J}_{3,c} \in W^{2,p/3}([t_0, \infty[), \quad \mathcal{H}_4 := \mathcal{J}_{3,p} + \mathcal{J}_{4,c} \in W^{2,p/4}([t_0, \infty[), \\ \mathcal{J}_{4,p} &\in W^{2,p/5}([t_0, \infty[).\end{aligned}$$

Then, we have that the functions  $\Theta_i$  and  $\Psi$  satisfying the requirements of the Theorem are defined as follows

$$\begin{array}{llll} p \in ]1, 2], & m = 1, & \Theta_k = \mathcal{H}_k \text{ for } k = 1, & \Psi = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{I}_{4,p} \\ p \in ]2, 3], & m = 2, & \Theta_k = \mathcal{H}_k \text{ for } k = 1, 2, & \Psi = \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{I}_{4,p}, \\ p \in ]3, 4], & m = 3, & \Theta_k = \mathcal{H}_k \text{ for } k = 1, 2, 3, & \Psi = \mathcal{H}_4 + \mathcal{I}_{4,p}, \\ p \in ]4, \infty[, & m = 4, & \Theta_k = \mathcal{H}_k \text{ for } k = 1, 2, 3, 4, & \Psi = \mathcal{I}_{4,p}, \end{array}$$

Thus, the result is valid for all  $p > 1$  and the Theorem is proved.  $\square$

#### 4. APPLICATIONS

##### 4.1. The Poincaré problem and Poincaré type result.

The fourth order linear differential equation of Poincaré type is given by

$$y^{(\text{iv})} + [a_3 + r_3(t)]y''' + [a_2 + r_2(t)]y'' + [a_1 + r_1(t)]y' + [a_0 + r_0(t)]y = 0, \quad (4.1)$$

where  $a_i$  are constants and  $r_i$  are real-valued functions. Note that (4.1) is a perturbation of the following constant coefficient equation:

$$y^{(\text{iv})} + a_3y''' + a_2y'' + a_1y' + a_0y = 0. \quad (4.2)$$

Now, let us consider the new variable  $z$  of the following type

$$z(t) = \frac{y'(t)}{y(t)} - \mu \quad \text{or equivalently} \quad y(t) = \exp \left( \int_{t_0}^t (z(s) + \mu) ds \right), \quad (4.3)$$

where  $y$  is a solution of (4.1) and  $\mu$  is an arbitrary root of the characteristic polynomial associated to (4.2). Then, differentiating  $y$  in (4.3) and replacing the results in (4.1), we deduce that  $z$  is a solution of the following third order nonlinear equation

$$\begin{aligned} & z''' + [4\mu + a_3]z'' + [6\mu^2 + 3a_3\mu + a_2]z' + [4\mu^3 + 3\mu^2a_3 + 2\mu a_2 + a_1]z \\ &= -[\mu^3r_3(t) + \mu^2r_2(t) + \mu r_1(t) + r_0(t)] - [3\mu^2r_3(t) + 2\mu r_2(t) + r_1(t)]z \\ &\quad - [3\mu r_3(t) + r_2(t)]z' - r_3(t)z'' - [12\mu + 3a_3 + 3r_3(t)]zz' - 4zz'' \\ &\quad - [6\mu^2 + 3\mu a_3 + a_2 + 3\mu r_3(t) + r_2(t)]z^2 - 3(z')^2 - 6z^2z' - [4\mu + r_3(t)]z^3 - z^4. \end{aligned} \quad (4.4)$$

Then, the analysis of original linear perturbed equation of fourth order (4.1) is translated to the analysis of a nonlinear third order equation (4.4).

We note that the equation (4.4) is of the type (1.1), since the constant coefficients  $b_i$  are

$$b_0 = 4\mu^3 + 3\mu^2a_3 + 2\mu a_2 + a_1, \quad b_1 = 6\mu^2 + 3\mu a_3 + a_2, \quad b_2 = 4\mu + a_3, \quad (4.5a)$$

and the functions  $\Omega_\alpha$  defining the coefficients of the polynomial  $\mathbb{R}$  are given by

$$\Omega_\alpha(t) = \begin{cases} -(\mu^3r_3(t) + \mu^2r_2(t) + \mu r_1(t) + r_0(t)), & \alpha = (0, 0, 0), \\ -(3\mu^2r_3(t) + 2\mu r_2(t) + r_1(t)), & \alpha = (1, 0, 0), \\ -(3\mu r_3(t) + r_2(t)), & \alpha = (0, 1, 0), \\ -r_3(t), & \alpha = (0, 0, 1), \\ -(12\mu + 3a_3 + r_3(t)), & \alpha = (1, 1, 0), \\ -4, & \alpha = (1, 0, 1), \\ -(6\mu^2 + 3\mu a_3 + a_2 + r_2(t) + 3\mu r_3(t)), & \alpha = (2, 0, 0), \\ -3, & \alpha = (0, 2, 0), \\ -6, & \alpha = (2, 1, 0), \\ -(4 + r_3(t)), & \alpha = (3, 0, 0), \\ -1, & \alpha = (4, 0, 0), \\ 0, & \text{otherwise.} \end{cases} \quad (4.5b)$$

Thus, we can apply the results of section 2.

**Theorem 4.1.** *Let us consider that the hypotheses*

$$(H_1) \quad \text{The set of characteristic roots for (4.2) is } \left\{ \lambda_i, i = \overline{1, 4} : \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 \right\} \subset \mathbb{R}.$$

(H<sub>2</sub>) Let  $p : \mathbb{R}^2 \rightarrow \mathbb{R}$  the function defined by

$$p(\mu, s) = -(\mu^3 r_3(t) + \mu^2 r_2(t) + \mu r_1(t) + r_0(t)). \quad (4.6)$$

The perturbation functions are selected such that  $\mathcal{G}(p(\lambda_i, \cdot))(t) \rightarrow 0$  and  $\mathcal{L}(r_j)(t) \rightarrow 0$ ,  $j = 0, 1, 2, 3$ , when  $t \rightarrow \infty$ , where  $\mathcal{G}$  and  $\mathcal{L}$  are the functionals defined on (2.7) and (2.8), respectively.

are satisfied. Then, for each  $i \in \{1, \dots, 4\}$ , the equation (4.4) with  $\mu = \lambda_i$  has a unique solution  $z_i$  such that  $z_i \in C_0^2([t_0, \infty[)$ .

*Proof.* The prove is reduced to apply the Theorem 2.1. Indeed, in the following we prove that hypotheses (P<sub>1</sub>) and (P<sub>2</sub>) are satisfied.

The hypothesis (H<sub>1</sub>) implies (P<sub>1</sub>). This result is a consequence of the following fact: if  $\lambda_i$  and  $\lambda_j$  are two distinct characteristic roots of the polynomial associated to (4.2), then  $\lambda_j - \lambda_i$  is a root of the characteristic polynomial associated with (4.4) when the right hand side is zero. Indeed, considering  $\lambda_i \neq \lambda_j$  satisfying the characteristic polynomial associated to (4.2), subtracting the equalities, dividing the result by  $\lambda_j - \lambda_i$  and using the identities

$$\begin{aligned} \lambda_j^3 + \lambda_j^2 \lambda_i + \lambda_j \lambda_i^2 + \lambda_i^3 &= (\lambda_j - \lambda_i)^3 + 4\lambda_i(\lambda_j - \lambda_i)^2 + 6\lambda_i^2(\lambda_j - \lambda_i) + 4\lambda_i^3 \\ a_3(\lambda_j^2 + \lambda_j \lambda_i + \lambda_i^2) &= a_3(\lambda_j - \lambda_i)^2 + 3a_3\lambda_i(\lambda_j - \lambda_i) + 3a_3\lambda_i^2 \\ a_2(\lambda_j - \lambda_i) &= a_2(\lambda_j - \lambda_i) + 2\lambda_i a_2, \end{aligned}$$

we deduce that  $\lambda_j - \lambda_i$  is a root of the characteristic polynomial associated to (4.4). Thus, if (H<sub>1</sub>) holds we have that

$$\text{for } i = 1 : 0 > \gamma_1 = \lambda_2 - \lambda_1 > \gamma_2 = \lambda_3 - \lambda_1 > \gamma_3 = \lambda_4 - \lambda_1 \quad (4.7a)$$

$$\text{for } i = 2 : \gamma_1 = \lambda_1 - \lambda_2 > 0 > \gamma_2 = \lambda_3 - \lambda_2 > \gamma_3 = \lambda_4 - \lambda_2 \quad (4.7b)$$

$$\text{for } i = 3 : \gamma_1 = \lambda_1 - \lambda_3 > \gamma_2 = \lambda_2 - \lambda_3 > 0 > \gamma_3 = \lambda_4 - \lambda_3 \quad (4.7c)$$

$$\text{for } i = 4 : \gamma_1 = \lambda_1 - \lambda_4 > \gamma_2 = \lambda_2 - \lambda_4 > \gamma_3 = \lambda_3 - \lambda_4 > 0 \quad (4.7d)$$

i.e.  $\gamma_1 > \gamma_2 > \gamma_3$  and (P<sub>1</sub>) is valid.

The hypothesis (H<sub>2</sub>) implies (P<sub>2</sub>). Indeed, by (4.5b) and (4.6) we note that

$$\begin{aligned} \mathcal{G}\left(\sum_{|\alpha|=0} \Omega_\alpha\right)(t) &= \mathcal{G}(-(\mu^3 r_3(\cdot) + \mu^2 r_2(\cdot) + \mu r_1(\cdot) + r_0(\cdot)))(t) \\ &= \mathcal{G}(p(\mu, \cdot))(t) \end{aligned} \quad (4.8)$$

$$\begin{aligned} \mathcal{L}\left(\sum_{|\alpha|=1} \Omega_\alpha\right)(t) &= \mathcal{L}(-((3\mu^2 + 3\mu + 1)r_3(\cdot) + (2\mu + 1)r_2(\cdot) + r_1(\cdot)))(t) \\ &\leq |3\mu^2 + 3\mu + 1|\mathcal{L}(r_3)(t) + |2\mu + 1|\mathcal{L}(r_2)(t) + \mathcal{L}(r_1)(t) \end{aligned} \quad (4.9)$$

$$\begin{aligned} \mathcal{L}\left(\sum_{|\alpha|=2} \Omega_\alpha\right)(t) &= \mathcal{L}(-((3\mu + 2)r_3(\cdot) + r_2(\cdot) + r_1(\cdot) + 3(1 + \mu)a_3 + a_2 + 6\mu^2 + 12\mu + 18))(t) \\ &\leq |3\mu + 2|\mathcal{L}(r_3)(t) + \mathcal{L}(r_2)(t) \\ &\quad + |3(1 + \mu)a_3 + a_2 + 6\mu^2 + 12\mu + 18|\mathcal{L}(1)(t). \end{aligned} \quad (4.10)$$

Thus, clearly if (H<sub>2</sub>) is satisfied, then (P<sub>2</sub>) is also satisfied.  $\square$

**Theorem 4.2.** Let us assume that the hypothesis (H<sub>1</sub>) and (H<sub>2</sub>) are satisfied. Denote by  $W[y_1, \dots, y_4]$  the Wronskian of  $\{y_1, \dots, y_4\}$ . Then, the equation (4.1) has a fundamental system of solutions given by

$$y_i(t) = \exp\left(\int_{t_0}^t [\lambda_i + z_i(s)]ds\right), \quad \text{with } z_i \text{ solution of (4.4) with } \mu = \lambda_i, \quad i \in \{1, 2, 3, 4\}. \quad (4.11)$$

Moreover the following properties about the asymptotic behavior

$$\frac{y_i'(t)}{y_i(t)} = \lambda_i, \quad \frac{y_i''(t)}{y_i(t)} = \lambda_i^2, \quad \frac{y_i'''(t)}{y_i(t)} = \lambda_i^3, \quad \frac{y_i^{(iv)}(t)}{y_i(t)} = \lambda_i^4, \quad (4.12)$$

$$W[y_1, \dots, y_4] = \prod_{1 \leq k < \ell \leq 4} (\lambda_\ell - \lambda_k) y_1 y_2 y_3 y_4 (1 + o(1)), \quad (4.13)$$

are satisfied when  $t \rightarrow \infty$ .

*Proof.* By Theorem 4.1 and change of variable (4.3), we have that the fundamental system of solutions for (4.1) is given by (4.11). Moreover, by (4.11) we deduce the identities

$$\frac{y_i'(t)}{y_i(t)} = [\lambda_i + z_i(t)] \quad (4.14)$$

$$\frac{y_i''(t)}{y_i(t)} = [\lambda_i + z_i(t)]^2 + z_i'(t), \quad (4.15)$$

$$\frac{y_i'''(t)}{y_i(t)} = [\lambda_i + z_i(t)]^3 + 3[\lambda_i + z_i(t)]z_i'(t) + z_i''(t), \quad (4.16)$$

$$\begin{aligned} \frac{y_i^{(iv)}(t)}{y_i(t)} &= [\lambda_i + z_i(t)]^4 + 6[\lambda_i + z_i(t)]^2 z_i'(t) + 3[z_i'(t)]^2 \\ &\quad + 4[\lambda_i + z_i(t)]z_i''(t) + z_i'''(t), \end{aligned} \quad (4.17)$$

Now, using the fact that  $z_i \in C_0^2([t_0, \infty[)$  is a solution of (4.11) with  $\mu = \lambda_i$ , we deduce the proof of (4.12). Moreover by the definition of the  $W[y_1, \dots, y_4]$ , some algebraic rearrangements and (4.12), we deduce (4.13).  $\square$

#### 4.2. Levison type theorem.

Let us introduce some notations. Consider the operators  $\mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3$  and  $\mathbb{F}_4$  defined as follows

$$\begin{aligned} \mathbb{F}_1(E)(t) &= \int_t^\infty e^{-(\lambda_2 - \lambda_1)(t-s)} |E(s)| ds, \\ \mathbb{F}_2(E)(t) &= \int_{t_0}^t e^{-(\lambda_1 - \lambda_2)(t-s)} |E(s)| ds + \int_t^\infty e^{-(\lambda_3 - \lambda_2)(t-s)} |E(s)| ds, \\ \mathbb{F}_3(E)(t) &= \int_{t_0}^t e^{-(\lambda_2 - \lambda_3)(t-s)} |E(s)| ds + \int_t^\infty e^{-(\lambda_4 - \lambda_3)(t-s)} |E(s)| ds, \\ \mathbb{F}_4(E)(t) &= \int_{t_0}^t e^{-(\lambda_3 - \lambda_4)(t-s)} |E(s)| ds; \end{aligned}$$

the positive numbers  $\sigma_i, A_i$  defined by

$$\begin{aligned} \sigma_i &= 3|\lambda_i|^2 + 5|\lambda_i| + 3 \\ &\quad + \left(19 + 7|\lambda_i| + |12\lambda_i + 3a_3| + |6\lambda_i^2 + 3\lambda_i a_3 + a_2|\right) \eta, \quad \eta \in ]0, 1/2[, \\ A_i &= \frac{1}{|\Upsilon_i|} \sum_{(j,k,\ell) \in I_i} |\lambda_k - \lambda_\ell| \left(1 + |\lambda_j - \lambda_i| + |\lambda_j - \lambda_i|^2\right), \end{aligned} \quad (4.18)$$

with

$$\begin{aligned} \Upsilon_i &= \prod_{k > j} (\lambda_k - \lambda_j), \quad k, j \in \{1, 2, 3, 4\} - \{i\}, \\ I_i &= \left\{ (j, k, \ell) \in \{1, 2, 3, 4\}^3 : (j, k, \ell) \neq (i, i, i), (k, \ell) \neq (j, j) \right\}; \end{aligned}$$

and define the sets

$$\mathcal{F}_i([t_0, \infty[) = \left\{ E : [t_0, \infty[ \rightarrow \mathbb{R} : \mathbb{F}_i(E)(t) \leq \rho_i := \min \left\{ \mathbb{F}_i(1)(t), \frac{1}{A_i \sigma_i} \right\} \right\}. \quad (4.19)$$

**Theorem 4.3.** Consider that the hypotheses  $(H_1), (H_2)$  on Theorem 4.1 are satisfied. Moreover consider that the following hypothesis:

$(H_3)$  Assume that the perturbation functions  $r_0, r_1, r_2, r_3 \in \mathcal{F}_i([t_0, \infty[)$ .

is satisfied. Then,  $z_i$ , the solution of (4.4) with  $\mu = \lambda_i$ , has the following asymptotic behavior

$$z_i(t), z'_i(t), z''_i(t) = \begin{cases} O\left(\int_t^\infty e^{-\beta(t-s)} |p(\lambda_1, s)| ds\right), & i = 1, \quad \beta \in ]\lambda_2 - \lambda_1, 0[, \\ O\left(\int_t^\infty e^{-\beta(t-s)} |p(\lambda_2, s)| ds\right), & i = 2, \quad \beta \in ]\lambda_3 - \lambda_2, 0[, \\ O\left(\int_{t_0}^\infty e^{-\beta(t-s)} |p(\lambda_3, s)| ds\right), & i = 3, \quad \beta \in ]\lambda_4 - \lambda_3, 0[, \\ O\left(\int_{t_0}^t e^{-\beta(t-s)} |p(\lambda_4, s)| ds\right), & i = 4, \quad \beta \in ]0, \lambda_3 - \lambda_4[, \end{cases} \quad (4.20)$$

where  $p$  is defined in (4.6).

*Proof.* By application of Theorem 2.2.  $\square$

**Theorem 4.4.** Let us assume that the hypotheses  $(H_1)$ ,  $(H_2)$  on Theorem 4.1 and hypothesis  $(H_3)$  on Theorem 4.3 are satisfied. Denote by  $\pi_i$  the number defined as follows

$$\pi_i = \prod_{k \in N_i} (\lambda_k - \lambda_i), \quad N_i = \{1, 2, 3, 4\} - \{i\}, \quad i = 1, \dots, 4,$$

and by  $p$  the function defined in (4.6). Then, the following asymptotic behavior

$$y_i(t) = e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1} \int_{t_0}^t [p(\lambda_i, s) + F(s, z_i(s), z'_i(s), z''_i(s))] ds\right), \quad (4.21)$$

$$y'_i(t) = (\lambda_i + o(1)) e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1} \int_{t_0}^t [p(\lambda_i, s) + F(s, z_i(s), z'_i(s), z''_i(s))] ds\right), \quad (4.22)$$

$$y''_i(t) = (\lambda_i^2 + o(1)) e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1} \int_{t_0}^t [p(\lambda_i, s) + F(s, z_i(s), z'_i(s), z''_i(s))] ds\right), \quad (4.23)$$

$$y'''_i(t) = (\lambda_i^3 + o(1)) e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1} \int_{t_0}^t [p(\lambda_i, s) + F(s, z_i(s), z'_i(s), z''_i(s))] ds\right), \quad (4.24)$$

$$y_i^{(iv)}(t) = (\lambda_i^4 + o(1)) e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1} \int_{t_0}^t [p(\lambda_i, s) + F(s, z_i(s), z'_i(s), z''_i(s))] ds\right), \quad (4.25)$$

holds, when  $t \rightarrow \infty$  with  $z_i, z'_i$  and  $z''_i$  given asymptotically by (4.20). Moreover, if  $r_0, r_1, r_2, r_3 \in L^1([t_0, \infty[)$  then the asymptotic forms

$$y_i(t) = e^{\lambda_i(t-t_0)} + o(1), \quad y'_i(t) = (\lambda_i + o(1)) e^{\lambda_i(t-t_0)}, \quad (4.26)$$

$$y''_i(t) = (\lambda_i^2 + o(1)) e^{\lambda_i(t-t_0)}, \quad y'''_i(t) = (\lambda_i^3 + o(1)) e^{\lambda_i(t-t_0)}, \quad (4.27)$$

$$y_i^{(iv)}(t) = (\lambda_i^4 + o(1)) e^{\lambda_i(t-t_0)}, \quad (4.28)$$

are satisfied when  $t \rightarrow \infty$ .

*Proof.* The proof of (4.21) follows from the identity

$$\begin{aligned} \int_{t_0}^t e^{-a\tau} \int_\tau^\infty e^{as} H(s) ds d\tau &= -\frac{1}{a} \left[ \int_t^\infty e^{-a(t-s)} H(s) ds - \int_{t_0}^\infty e^{-a(t_0-s)} H(s) ds \right] \\ &\quad + \frac{1}{a} \int_{t_0}^t H(\tau) d\tau \end{aligned} \quad (4.29)$$

and from (2.10)-(2.11). Now, we develop the proof for  $i = 1$ . Indeed, by (4.11) we have that

$$y_1(t) = \exp\left(\int_{t_0}^t (\lambda_1 + z_1(\tau)) d\tau\right) = e^{\lambda_1(t-t_0)} \exp\left(\int_{t_0}^t z_1(\tau) d\tau\right). \quad (4.30)$$

By (2.10)-(2.11), (2.22), (4.7a), (4.29), and the fact that  $\pi_1 = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)$ , we have that

$$\begin{aligned}
\int_{t_0}^t z_1(\tau) d\tau &= \frac{1}{\Upsilon_1} \int_{t_0}^t \int_{t_0}^\infty g_1(\tau, s) \left( p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s)) \right) d\tau ds \\
&= \frac{1}{\Upsilon_1} \int_{t_0}^t \int_\tau^\infty \left[ (\lambda_4 - \lambda_3) e^{-(\lambda_2 - \lambda_1)(\tau - s)} + (\lambda_2 - \lambda_4) e^{-(\lambda_3 - \lambda_1)(\tau - s)} \right. \\
&\quad \left. + (\lambda_3 - \lambda_2) e^{-(\lambda_4 - \lambda_1)(\tau - s)} \right] \left( p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s)) \right) d\tau ds \\
&= \frac{1}{\Upsilon_1} \left[ \frac{\lambda_4 - \lambda_3}{\lambda_2 - \lambda_1} + \frac{\lambda_2 - \lambda_4}{\lambda_3 - \lambda_1} + \frac{\lambda_3 - \lambda_2}{\lambda_4 - \lambda_1} \right] \int_{t_0}^t \left( p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s)) \right) ds \\
&\quad + \frac{1}{\Upsilon_1} \left[ \frac{\lambda_4 - \lambda_2}{\lambda_2 - \lambda_1} \left\{ \int_t^\infty e^{-(\lambda_2 - \lambda_1)(t - s)} \left( p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s)) \right) ds \right. \right. \\
&\quad \left. \left. - \int_{t_0}^\infty e^{-(\lambda_2 - \lambda_1)(t_0 - s)} \left( p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s)) \right) ds \right\} \right. \\
&\quad \left. + \frac{1}{\Upsilon_1} \left[ \frac{\lambda_2 - \lambda_4}{\lambda_3 - \lambda_1} \left\{ \int_t^\infty e^{-(\lambda_3 - \lambda_1)(t - s)} \left( p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s)) \right) ds \right. \right. \right. \\
&\quad \left. \left. - \int_{t_0}^\infty e^{-(\lambda_3 - \lambda_1)(t_0 - s)} \left( p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s)) \right) ds \right\} \right. \right. \\
&\quad \left. \left. + \frac{1}{\Upsilon_1} \left[ \frac{\lambda_3 - \lambda_2}{\lambda_4 - \lambda_1} \left\{ \int_t^\infty e^{-(\lambda_4 - \lambda_1)(t - s)} \left( p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s)) \right) ds \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \int_{t_0}^\infty e^{-(\lambda_4 - \lambda_1)(t_0 - s)} \left( p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s)) \right) ds \right\} \right] \right] \right. \\
&= \frac{1}{\pi_1} \int_{t_0}^t \left( p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s)) \right) ds + o(1)
\end{aligned}$$

Then, (4.21) is valid for  $i = 1$ . The proof of (4.21) for  $i = 2, 3, 4$  is analogous. Now the proof of (4.22) follows by (4.21) and (4.14)-(4.17).

To prove (4.26)-(4.28) we apply the decomposition of Theorem 3.1.  $\square$

### 4.3. Hartman-Wintner and Harris-Lutz type theorems.

By application of Theorem 3.1 we get the following results.

**Theorem 4.5.** (*Hartman-Wintner*) *Let us assume that the hypotheses  $(H_1)$ ,  $(H_2)$  on Theorem 4.1 and hypothesis  $(H_3)$  on Theorem 4.3 are satisfied. If  $r_0, r_1, r_2, r_3 \in L^1([t_0, \infty[)$  for  $p \in ]1, 2]$ , then the asymptotic behavior (4.26)-(4.28) is valid when  $t \rightarrow \infty$ .*

**Theorem 4.6.** (*Harris-Lutz*) *Let us assume that the hypotheses  $(H_1)$ ,  $(H_2)$  on Theorem 4.1 and hypothesis  $(H_3)$  on Theorem 4.3 are satisfied. If  $r_0, r_1, r_2, r_3 \in L^p([t_0, \infty[)$  for  $p \geq 1$ , then the following asymptotic behavior*

$$y_i(t) = e^{\lambda_i(t-t_0)} \exp \left( \int_{t_0}^t \left[ \sum_{k=1}^m \Theta_k(s) + \Psi(s) \right] ds \right), \quad (4.31)$$

$$y_i'(t) = \left( \lambda_i + o(1) \right) e^{\lambda_i(t-t_0)} \exp \left( \int_{t_0}^t \left[ \sum_{k=1}^m \Theta_k'(s) + \Psi'(s) \right] ds \right), \quad (4.32)$$

$$y_i''(t) = \left( \lambda_i^2 + o(1) \right) e^{\lambda_i(t-t_0)} \exp \left( \int_{t_0}^t \left[ \sum_{k=1}^m \Theta_k''(s) + \Psi''(s) \right] ds \right), \quad (4.33)$$



$$y_i'''(t) = \left(\lambda_i^3 + o(1)\right) e^{\lambda_i(t-t_0)} \exp\left(\int_{t_0}^t \left[\sum_{k=1}^m \Theta_k'''(s) + \Psi'''(s)\right] ds\right), \quad (4.34)$$

$$y_i^{(\text{iv})}(t) = \left(\lambda_i^4 + o(1)\right) e^{\lambda_i(t-t_0)} \exp\left(\int_{t_0}^t \left[\sum_{k=1}^m \Theta_k^{(\text{iv})}(s) + \Psi^{(\text{iv})}(s)\right] ds\right), \quad (4.35)$$

holds, when  $t \rightarrow \infty$ , where  $m, \Theta_i, \psi$  is the notation introduced on Theorem 3.1.

**4.4. Unbounded coefficients.** Let us consider the following equation

$$y^{(\text{iv})}(t) - 2[q(t)]^{1/2} y'''(t) - q(t) y''(t) + 2[q(t)]^{3/2} y'(t) + r(t) y(t) = 0, \quad (4.36)$$

where  $q$  and  $r$  are given functions such that

$$q(t) \rightarrow \infty, \quad \text{when} \quad t \rightarrow \infty. \quad (4.37)$$

Then, we note that (4.36) is not of a Poincaré type. However, we can apply the results of asymptotic behavior Theorems 4.2, 2.2 and 4.4, after an appropriate transformation. Indeed, let us consider the following change of variable

$$z(s) = y(s)[q(s)]^{1/4} \quad \text{with} \quad s = \int_{t_0}^t [q(\tau)]^{1/2} d\tau. \quad (4.38)$$

Note that  $ds(t) = [q(t)]^{1/2} dt$ . Then, we can rewrite (4.36) in an equivalent way as follows

$$z^{(\text{iv})} + (-2 + r_3(t)) z''' + (-1 + r_2(t)) z'' + (2 + r_1(t)) z' + r_0(t) z = 0, \quad (4.39)$$

where

$$\begin{aligned} r_0(t) = & -\frac{q^{(\text{iv})}(t)}{4q(t)} + \frac{q'''(t)}{2q(t)} - \frac{5q'''(t)}{8[q(t)]^{5/2}} + \frac{5q'''(t)q'(t)}{4[q(t)]^2} + \left(\frac{3}{8[q(t)]^3} + \frac{15}{16[q(t)]^2}\right) [q''(t)]^2 \\ & - \left(\frac{15}{8[q(t)]^2} - \frac{1}{8[q(t)]^{7/2}} - \frac{1}{4[q(t)]^3}\right) q''(t)q'(t) + \frac{q''(t)}{4q(t)} \\ & - \left(\frac{135}{32[q(t)]^3} - \frac{5}{4[q(t)]^{7/2}} - \frac{11}{16[q(t)]^4} - \frac{3}{16[q(t)]^{9/2}}\right) q''(t)[q'(t)]^2 - \frac{q'(t)}{2q(t)} \\ & + \left(\frac{1}{8[q(t)]^{5/2}} - \frac{5}{16[q(t)]^2}\right) [q'(t)]^2 + \left(\frac{45}{32[q(t)]^3} - \frac{15}{16[q(t)]^{7/2}} - \frac{1}{8[q(t)]^4}\right) [q'(t)]^3 \\ & + \left(\frac{585}{256[q(t)]^4} - \frac{135}{64[q(t)]^{9/2}} - \frac{5}{64[q(t)]^5} - \frac{3}{32[q(t)]^{11/2}}\right) [q'(t)]^4 + \frac{r(t)}{[q(t)]^2}, \end{aligned} \quad (4.40)$$

$$\begin{aligned} r_1(t) = & \left(\frac{1}{2[q(t)]^{5/2}} - \frac{1}{q(t)}\right) q'''(t) + \left(\frac{3}{2q(t)} - \frac{1}{[q(t)]^2}\right) q''(t) \\ & + \left(\frac{15}{4[q(t)]^2} - \frac{17}{8[q(t)]^{5/2}} - \frac{1}{[q(t)]^3} - \frac{3}{4[q(t)]^{7/2}}\right) q''(t)q'(t) \\ & - \left(\frac{45}{16[q(t)]^3} + \frac{45}{16[q(t)]^{7/2}} - \frac{3}{8[q(t)]^{9/2}} - \frac{1}{8[q(t)]^4}\right) [q'(t)]^3 \\ & - \left(\frac{15}{8[q(t)]^2} - \frac{3}{2[q(t)]^{5/2}} - \frac{1}{2[q(t)]^3}\right) [q'(t)]^2 + \left(\frac{1}{2q(t)} - \frac{1}{2[q(t)]^{3/2}}\right) q'(t) \\ & - \frac{17}{8[q(t)]^{5/2}}, \end{aligned} \quad (4.41)$$

$$\begin{aligned} r_2(t) = & \left(\frac{1}{2[q(t)]^4} + \frac{3}{2[q(t)]^2} - \frac{3}{2q(t)}\right) q''(t) \\ & + \left(\frac{15}{16[q(t)]^2} - \frac{3}{4[q(t)]^{5/2}} - \frac{1}{4[q(t)]^3} - \frac{3}{8[q(t)]^4}\right) [q'(t)]^2 \\ & - \left(\frac{9}{8[q(t)]^{5/2}} + \frac{15}{16[q(t)]^2} + \frac{3}{[q(t)]^{3/2}} - \frac{3}{2q(t)}\right) q'(t), \end{aligned} \quad (4.42)$$

$$r_3(t) = \left( \frac{3}{[q(t)]^{3/2}} - \frac{1}{4q(t)} \right) q'(t) - \frac{3}{4q(t)}. \quad (4.43)$$

Thus, by application of Theorem 4.3 we get the following theorem.

**Theorem 4.7.** *Let us consider that  $q$  and  $r$  are two functions such that the functions*

$$\begin{aligned} & \left( \frac{q'}{q} \right)^{2k} \text{ for } k = 1, \dots, 4; \quad \left( \frac{q''}{q} \right)^{2k} \text{ for } k = 1, 2; \quad \frac{(q'')^2 (q')^{2(k+1)}}{q^{2(k+2)}} \text{ for } k = 0, 1; \\ & \frac{r^2}{q^4}; \quad \frac{q''' q'}{q^2}; \quad \left( \frac{q'''}{q} \right)^2 \text{ and } \left( \frac{q^{(iv)}}{q} \right)^2 \end{aligned}$$

are in  $L^1([t_0, \infty))$ . Moreover assume that  $q \in C^4([t_0, \infty))$  is an increasing function on  $[t_0, \infty)$  and satisfies (4.37). Then the equation (4.36) asymptotically has the following fundamental system of solutions

$$\begin{aligned} y_i(t) &= [q(t)]^{-1/4} (1 + o(1)) \\ &\times \begin{cases} \exp \left( - \int_{t_0}^t [q(\tau)]^{1/2} d\tau + \frac{1}{6} \int_{t_0}^t (r_0 - r_1 + r_2 - r_3)(\tau) [q(\tau)]^{1/2} d\tau \right), & i = 1, \\ \exp \left( - \frac{1}{2} \int_{t_0}^t [q(\tau)]^{1/2} r_0(\tau) d\tau \right), & i = 2, \\ \exp \left( \int_{t_0}^t [q(\tau)]^{1/2} d\tau + \frac{1}{2} \int_{t_0}^t (r_0 + r_1 + r_2 + r_3)(\tau) [q(\tau)]^{1/2} d\tau \right), & i = 3, \\ \exp \left( 2 \int_{t_0}^t [q(\tau)]^{1/2} d\tau - \frac{1}{6} \int_{t_0}^t (r_0 + 2r_1 + 4r_2 + 8r_3)(\tau) [q(\tau)]^{1/2} d\tau \right), & i = 4, \end{cases} \end{aligned} \quad (4.44)$$

where  $r_i$  are defined on (4.40)-(4.43).

## 5. EXAMPLES

In this section we consider three examples. In the first two examples the classical results of Levinson, Hartman-Witner and Harris-Lutz cannot be applied but we can apply the Theorem 4.4. In the third example we present an application of Theorem 4.7.

**5.1. Example 1.** In this example we present a case where the results Levinson type or Hartman-Witner type or Harris-Lutz type cannot be applied. Indeed, let us consider the differential equation

$$y^{(4)} + 2y^{(3)} + 13y^{(2)} - 14y^{(1)} + \left[ \frac{3}{t^{1/(p+1)}(\sin t + 2)} + 24 \right] y = 0, \quad p \geq 1, \quad t \in [1, +\infty[. \quad (5.1)$$

We note that  $r_0(t) = 3t^{-1/p+1} (\sin t + 2)^{-1} \notin L^p([1, +\infty[)$  for any  $p \geq 1$ . Then, the classical generalizations of Poincaré type theorems: the Levinson and the Hartman-Wintner theorems, can not be applied to obtain the asymptotic behavior of (5.1). Now, in order to apply the Theorem 4.4, we have that

- (a) The set of characteristic roots of the linear no perturbed equation associated to (4.1), i.e.  $y^{(4)} + 2y^{(3)} + 13y^{(2)} - 14y^{(1)} + 24y = 0$ , is given by  $\{3, 1, -2, -4\}$ . Then (H1) is satisfied.
- (b) We note that  $r_0(t) \rightarrow 0$  when  $t \rightarrow \infty$  which implies that  $\mathfrak{L}(r_0)(t) \rightarrow 0$  when  $t \rightarrow \infty$  and  $\mathcal{L}(r_1)(t) = \mathcal{L}(r_2)(t) = \mathcal{L}(r_0)(t) = 0$  since  $r_1 = r_2 = r_3 = 0$ . Moreover  $p(\lambda_i, s) = (r_0)(s)$  and as  $(p(\lambda_i, \cdot))(t) = \mathcal{G}(r_0)(t) \leq \mathcal{L}(r_0)(t)$  we deduced that  $\mathcal{G}(p(\lambda_i, \cdot))(t) \rightarrow 0$  when  $t \rightarrow \infty$ . Thus, (H2) is also satisfied.
- (c) We note that

$$\begin{aligned} \mathbb{F}_1 &= \frac{1}{2}, \quad \mathbb{F}_2 = \frac{1}{2}[1 - e^{-2(t-1)}] + \frac{1}{3}, \quad \mathbb{F}_3 = \frac{1}{3}[1 - e^{-3(t-1)}] + \frac{1}{2}, \quad \mathbb{F}_4 = \frac{1}{2}[1 - e^{-2(t-1)}], \\ \sigma_1 &= 45 + 167\eta, \quad \sigma_2 = 11 + 69\eta, \quad \sigma_3 = 25 + 76\eta, \quad \sigma_4 = 61 + 174\eta, \\ A_1 &= \frac{34}{3}, \quad A_2 = \frac{26}{7}, \quad A_3 = \frac{26}{7}, \quad A_4 = \frac{34}{3}. \end{aligned}$$

Then, the sets  $\mathcal{F}_i([1, \infty[)$  given in given (3.19) are well defined. We note that, naturally,  $r_1 = r_2 = r_3 = 0 \in \mathcal{F}_i$ . Moreover, from  $r_0(t) \rightarrow 0$  when  $t \rightarrow \infty$  we can prove that  $\mathbb{F}_i(r_0(t)) \rightarrow 0$  when  $t \rightarrow \infty$ . Then, we have that  $r_0 \in \mathcal{F}_i([1, \infty[)$ . Hence, (H3) is satisfied.

Thus, from (a) – (c), we can apply the Theorem 4.4 and the asymptotic formulas are given by

$$\begin{aligned} y_1(t) &= e^{3(t-s)} \exp \left\{ \left( \frac{1}{30} \int_1^t \left[ \frac{6}{s^{\frac{1}{p+1}} [\sin s + 2]} + f_1(s) \right] ds \right) \right\}, \\ y_2(t) &= e^{(t-s)} \exp \left\{ \left( \frac{-1}{70} \int_1^t \left[ \frac{6}{s^{\frac{1}{p+1}} [\sin s + 2]} + f_2(s) \right] ds \right) \right\}, \\ y_3(t) &= e^{-2(t-s)} \exp \left\{ \left( \frac{1}{70} \int_1^t \left[ \frac{6}{s^{\frac{1}{p+1}} [\sin s + 2]} - f_3(s) \right] ds \right) \right\}, \\ y_4(t) &= e^{-4(t-s)} \exp \left\{ \left( \frac{-1}{30} \int_1^t \left[ \frac{6}{s^{\frac{1}{p+1}} [\sin s + 2]} + f_4(s) \right] ds \right) \right\}, \end{aligned}$$

where

$$\begin{aligned} f_1(t) &= -42z_1 z_1' + 4z_1 z_1'' + 265z_1^2 + 3(z_1')^2 + 6z_1^2 z_1' + 24z_1^3 + z_1^4, \\ f_2(t) &= -18z_2 z_2' + 4z_2 z_2'' + 25z_2^2 + 3(z_2')^2 + 6z_2^2 z_2' + 4z_2^3 + z_2^4, \\ f_3(t) &= -18z_3 z_3' + 4z_3 z_3'' + 25z_3^2 + 3(z_3')^2 + 6z_3^2 z_3' - 8z_3^3 + z_3^4, \\ f_4(t) &= -42z_4 z_4' + 4z_4 z_4'' + 84z_4^2 + 3(z_4')^2 + 6z_4^2 z_4' - 16z_4^3 + z_4^4, \end{aligned}$$

and  $z_i(t)$  satisfies the following asymptotic behavior

$$z_i(t), z_i'(t), z_i''(t) = \begin{cases} O\left(\int_t^\infty \frac{3e^{-\beta(t-s)}}{s^{\frac{1}{p+1}} [\sin s + 2]} ds\right), & i = 1, \quad \beta \in [-2, 0[, \\ O\left(\int_1^\infty \frac{3e^{-\beta(t-s)}}{s^{\frac{1}{p+1}} [\sin s + 2]} ds\right), & i = 2, \quad \beta \in [-3, 0[, \\ O\left(\int_1^\infty \frac{3e^{-\beta(t-s)}}{s^{\frac{1}{p+1}} [\sin s + 2]} ds\right), & i = 3, \quad \beta \in [-2, 0[, \\ O\left(\int_1^t \frac{3e^{-\beta(t-s)}}{s^{\frac{1}{p+1}} [\sin s + 2]} ds\right), & i = 4, \quad \beta \in ]0, 7], \end{cases}$$

**5.2. Example 2.** Here we present an example with a perturbation function such that the results of Levinson, Hartman-Witner, Harris-Lutz or Eastham types cannot be applied. Indeed, let us consider the differential equation

$$y^{(4)} + 10y^{(3)} + 35y^{(2)} + 50y^{(1)} + \left[ \frac{3}{(\cos t + 2) \log t} + 24 \right] y = 0, \quad t \in [2, +\infty[. \quad (5.2)$$

We note that

$$r_0(t) = \left[ \frac{3}{\log t [\cos t + 2]} \right] \notin L^p([2, +\infty[), \text{ for any } p \geq 1$$

Then, the classical generalizations of Poincaré type theorems, the Levinson theorem, the Hartman-Wintner can not be applied to obtain the asymptotic behavior of the solutions for (5.2). Moreover, we note that the inequality

$$\frac{|\sin t|}{9 \log t} \leq \frac{|\sin t|}{(\cos t + 2)^2 \log t}$$

is valid on  $[2, +\infty[$ , which implies, by the comparison criteria and since  $\sin t / \log t [\cos t + 2]^2 \notin L^1([2, +\infty[)$ , the fact that

$$\frac{\sin t}{\log t [\cos t + 2]^2} \notin L^1([2, +\infty[).$$

Then, we can deduce that

$$r_0'(t) = 3 \left[ \frac{1}{t(\log t)^2 [\cos t + 2] \ln 10} + \frac{\sin t}{\log t [\cos t + 2]^2} \right] \notin L^1([2, +\infty[).$$

Thus, we can not apply the classic theorem of Eastham. However, we note that

- (a) The set of characteristic roots of the linear no perturbed equation associated to (5.2), i.e.  $y^{(4)} + 10y^{(3)} + 35y^{(2)} + 50y^{(1)} + 24y = 0$ , is given by  $\{-1, -2, -3, -4\}$ . Then (H1) is satisfied.
- (b) We note that  $r_0(t) \rightarrow 0$  when  $t \rightarrow \infty$  which implies that  $\mathcal{L}(r_0)(t) \rightarrow 0$  when  $t \rightarrow \infty$  and  $\mathcal{L}(r_1)(t) = \mathfrak{L}(r_2)(t) = \mathcal{L}(r_0)(t) = 0$  since  $r_1 = r_2 = r_3 = 0$ . Moreover  $p(\lambda_i, s) = (r_0)(s)$  and as  $\mathcal{G}(p(\lambda_i, \cdot))(t) = \mathcal{G}(r_0)(t) \leq \mathcal{L}(r_0)(t)$  we deduced that  $\mathcal{G}(p(\lambda_i, \cdot))(t) \rightarrow 0$  when  $t \rightarrow \infty$ . Thus, (H2) is also satisfied.
- (c) We note that

$$\begin{aligned} \mathbb{F}_1 &= 1, & \mathbb{F}_2 &= 2 - e^{(t-2)}, & \mathbb{F}_3 &= 2 - e^{-(t-2)}, & \mathbb{F}_4 &= 1 - e^{-(t-2)}, \\ \sigma_1 &= 11 + 55\eta, & \sigma_2 &= 25 + 38\eta, & \sigma_3 &= 45 + 47\eta, & \sigma_4 &= 71 + 76\eta, \\ A_1 &= \frac{29}{2}, & A_2 &= \frac{13}{3}, & A_3 &= \frac{13}{3}, & A_4 &= \frac{29}{2}. \end{aligned}$$

Then, the sets  $\mathcal{F}_i([1, \infty[)$  given in given (3.19) are well defined. We note that, naturally,  $r_1 = r_2 = r_3 = 0 \in \mathcal{F}_i$ . Moreover, from  $r_0(t) \rightarrow 0$  when  $t \rightarrow \infty$  we can prove that  $\mathbb{F}_i(r_0(t)) \rightarrow 0$  when  $t \rightarrow \infty$ . Then, we have that  $r_0 \in \mathcal{F}_i([1, \infty[)$ . Hence, (H3) is satisfied.

Thus, from (a) – (c), we can apply the Theorem 4.4 and the asymptotic formulas are given by

$$\begin{aligned} y_1(t) &= e^{-(t-s)} \exp \left\{ \left( \frac{1}{6} \int_1^t \left[ \frac{6}{(\cos s + 2) \log s} + f_1(s) \right] ds \right) \right\}, \\ y_2(t) &= e^{-2(t-s)} \exp \left\{ \left( \frac{-1}{2} \int_1^t \left[ \frac{6}{(\cos s + 2) \log s} + f_2(s) \right] ds \right) \right\}, \\ y_3(t) &= e^{-3(t-s)} \exp \left\{ \left( \frac{1}{2} \int_1^t \left[ \frac{6}{(\cos s + 2) \log s} - f_3(s) \right] ds \right) \right\}, \\ y_4(t) &= e^{-4(t-s)} \exp \left\{ \left( \frac{-1}{6} \int_1^t \left[ \frac{6}{(\cos s + 2) \log s} - f_4(s) \right] ds \right) \right\}, \end{aligned}$$

where

$$\begin{aligned} f_1(t) &= 18z_1 z_1' + 4z_1 z_1'' + 6z_1^2 + 3(z_1')^2 + 6z_1^2 z_1' - 4z_1^3 + z_1^4, \\ f_2(t) &= 6z_2 z_2' + 4z_2 z_2'' - z_2^2 + 3(z_2')^2 + 6z_2^2 z_2' - 8z_2^3 + z_2^4, \\ f_3(t) &= -6z_3 z_3' + 4z_3 z_3'' - z_3^2 + 3(z_3')^2 + 6z_3^2 z_3' - 12z_3^3 + z_3^4, \\ f_4(t) &= -18z_4 z_4' + 4z_4 z_4'' + 11z_4^2 + 3(z_4')^2 + 6z_4^2 z_4' - 16z_4^3 + z_4^4, \end{aligned}$$

and  $z_i(t)$  satisfies the following asymptotic behavior

$$z_i(t), z_i'(t), z_i''(t) = \begin{cases} O\left(\int_t^\infty \frac{3e^{-\beta(t-s)}}{(\cos s + 2) \log s} ds\right), & i = 1, \quad \beta \in [-1, 0[, \\ O\left(\int_1^\infty \frac{3e^{-\beta(t-s)}}{(\cos s + 2) \log s} ds\right), & i = 2, \quad \beta \in [-1, 0[, \\ O\left(\int_1^\infty \frac{3e^{-\beta(t-s)}}{(\cos s + 2) \log s} ds\right), & i = 3, \quad \beta \in [-1, 0[, \\ O\left(\int_1^t \frac{3e^{-\beta(t-s)}}{(\cos s + 2) \log s} ds\right), & i = 4, \quad \beta \in ]0, 1]. \end{cases}$$

**5.3. Example 3.** Let us consider the following equation

$$y^{(\text{iv})}(t) - 2t^{\alpha/2} y'''(t) - t^\alpha y''(t) + 2t^{\alpha/2} y'(t) + y(t) = 0, \quad \alpha > 0, \quad (5.3)$$

i.e. the equation (4.36) with  $q(t) = t^\alpha$  and  $r(t) = 1$ . The coefficients are unbounded and the classical results for the asymptotic behavior cannot be applied. However, we note that

$$\left( \frac{q'(t)}{q(t)} \right)^{2k} = \left( \frac{\alpha}{t} \right)^{2k} \text{ for } k = 1, \dots, 4; \quad \left( \frac{q''(t)}{q(t)} \right)^{2k} = \left( \frac{\alpha(\alpha-1)}{t^2} \right)^{2k} \text{ for } k = 1, 2;$$

$$\begin{aligned} \frac{(q''(t))^2(q'(t))^{2(k+1)}}{q(t)^{2(k+2)}} &= \frac{\alpha^{2k+4}(\alpha-1)^2}{t^{2(k+3)}} \text{ for } k = 0, 1; \quad \frac{r(t)^2}{q(t)^4} = \frac{1}{t^{4\alpha}}; \\ \frac{q'''(t)q'(t)}{q(t)^2} &= \frac{\alpha(\alpha-1)(\alpha-2)}{t^4}; \quad \left(\frac{q'''(t)}{q(t)}\right)^2 = \frac{\alpha^2(\alpha-1)(\alpha-2)^2}{t^6} \quad \text{and} \\ \left(\frac{q^{(iv)}}{q}\right)^2 &= \frac{\alpha^2(\alpha-1)^2(\alpha-2)^2(\alpha-3)^2}{t^8}. \end{aligned}$$

Then the hypotheses of Theorem 4.7 are satisfied and the equation (5.3) has a fundamental system of solutions given by (4.44) with  $q(t) = t^\alpha$ .

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